Sys Rev Pharm 2020;11(6):1302-1310
A multifaceted review journal in the field of pharmacy

The Aspects and Stages of Reversible Thinking of Secondary School Students in Resolving the Problems of Fractional Numbers

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ABSTRACT

This study aimed at (1) determining the aspects of students' reversible thinking skills in solving the problems of fractional numbers, and (2) determining the stages of students' reversible thinking in solving the problems of fractional numbers. This study belongs to descriptive qualitative research. The subjects of this study were two students of the seventh grade of SMPN 3 Malang, Indonesia. The data was gathered through written tests, observations, and interviews. The results showed that the aspects of students' reversible thinking were verbally negated, physically negated, reciprocity through conversion, reciprocity caused by negation (verbal and physical), and the negation occurred, one of them, was due to the presence of inverse principle. These results also showed that negation is one of the ways to overcome students' perplexity in solving problems. Based on the stages of students' reversible thinking, the second subject shows the importance of thinking reversibly in solving problems, especially in the field of mathematics.

Keywords: Reversible Thinking, Negation, Reciprocity, Fraction Problems

INTRODUCTION

Reversible thinking is the reconstruction of single-unit of a part or mixed fraction (Tunç-Pekkan, 2015). Maf'ulah, et. al (2017) identify the aspects of reversible thinking of elementary school students are negation and reciprocity. Maf'ulah, et. al (2017) identify the negation because it uses an inverse, which is canceling the addition of 8 $(+8)$ by subtracting 8 where the subtraction operation is the inverse of the subtraction operation. Negation imposes the idea that every direct operation has its opposition (Ramful, 2014). Meanwhile, Maf'ulah, et. al (2017) identify that the reciprocity of elementary school students is using compensation, which is to divide both sides of the initial equation with 4. Ramful (2014) states that Piaget uses reciprocity in which it refers to coordination between two sides of a relation. For example, if $A > B$, then $B < A$ or if $A + B = C$, then $A = C - B$ or if $A = B$, then $B = A$.

Negation and reciprocity are two types of reversibility. In a concrete problem, Hackenberg (2010) illustrates reversibility based on Piaget's theory through a balance. First, put an object on one side of the balance and then take it back from its position so that the balance of the scale returns to its initial position. This is an example of inverse or negation. Second, put an object on one balanced side and then put another object with the same weight on the other side so that it is balanced. This is an example of compensation. This shows that negation and reciprocity do not occur simultaneously on concrete problems. While on the other hand, it is formal, Maf'ulah, et. al (2017) show that canceling $+8$ with -8 from the equation $\sqrt{\frac{24+a}{4}} + 8 = 10$ so that it becomes $\sqrt{\frac{24+a}{4}} = 10 - 8$ is the principle of negation. Basically, the result of negation will cause that if $\sqrt{\frac{24+a}{4}} = 10 - 8$, then $\sqrt{\frac{24+a}{4}} = 2$, and this is the principle of reciprocity.

A child is said to be able to think reversibly with negation if (1) the child is able to express verbally, related to denial (Nordmeyer & Frank, 2014); (2) the child is able to cancel operations, such as +1 – 1 (Ramful & Olive, 2008); and (3) the child is able to return an operation to the initial condition (Saparwadi et al., 2019). Meanwhile, a child is said to be able to think reversibly with reciprocity if the child is able to compare two conditions in a balanced way, such as operating the initial equation into the next equation (Maf'ulah et al., 2017; Ramful, 2014, 2015; Ramful & Olive, 2008), so that the two equations are equivalent. Based on these characteristics, a child is able to think reversibly if a child is able to express a condition verbally related to the raised denial, able to carry out cancellation operations, able to return an operation to the initial conditions of the actions that have occurred, and able to compare or equalize two conditions, both the initial condition and the final one. The basic topics of mathematics used in knowing students' reversible thinking skills based on previous research are the concept of fractions through single-unit reconstruction with several fractions (Tunç-Pekkan, 2015) and algebraic equations (Maf'ulah et al., 2017).

One of the challenging mathematics materials of reversible thinking from previous studies (Maf'ulah et al., 2017; Tunç-Pekkan, 2015) is fractional numbers. Some researchers (Alghazo & Alghazo, 2017; Bottge et al., 2014; Malone & Fuchs, 2017; Sa'dijah et al., 2017; Sa'dijah, 2010, 2011, 2013, 2014; Sa'dijah et al., 2018; Sa'dijah et al., 2016) have found that students have difficulties in the concept of fractions. The material of fractional numbers has been studied by several researchers (Hackenberg, 2013; Hackenberg & Lee, 2016; Lee & Hackenberg, 2014; Ramful, 2014; Ramful & Olive, 2008) in reversible situations, and still not focusing on students' reversible thinking. Meanwhile, the research conducted by Tunç-Pekkan (2015) about fractional numbers only indicates the occurrence of reversible thinking.

Some studies conducted by several researchers (Maf'ulah et al., 2017; Tunç-Pekkan, 2015) about reversible thinking is still limited to elementary school students or children who are in the concrete operational stage and are still in the developmental stage of reversible thinking. Meanwhile, reversible thinking of students at the stage of formal operational development has not been investigated. Therefore, it is important to study further how students at the formal operational stage think reversibly. To fulfill the gap, this study examined the reversible thinking skill of secondary school students in the material of fractional numbers. The main focus of this study was what are the aspects and stages of reversible thinking of secondary school students or 13-year-old children in solving the problems of fractional numbers. Based on preliminary observations, 13-year-old children are still dominated by secondary school students who still have difficulties in using reversible thinking skills in solving the problems of fractional numbers presented in the form of narration. This argument was also affirmed by one of the secondary school mathematics teachers, Anita Yulita, that most students at the secondary school level had difficulty with narration problem. This can be seen from 120 students who were given algebraic problems in the form of narrative questions (see Figure 1) and the results of the student's answer (see Figure 2) as follows.

Figure 1. Algebraic problems

Ian has 12 dozen books. The number of Adi and Rohma's books is as many as of *Ian* and Lina's books. Meanwhile, the number of Adi's books is ¾ of Lina's, and the number of Rohma's books is 3 dozen less than the number of Lina's books. How many books does Lina have?

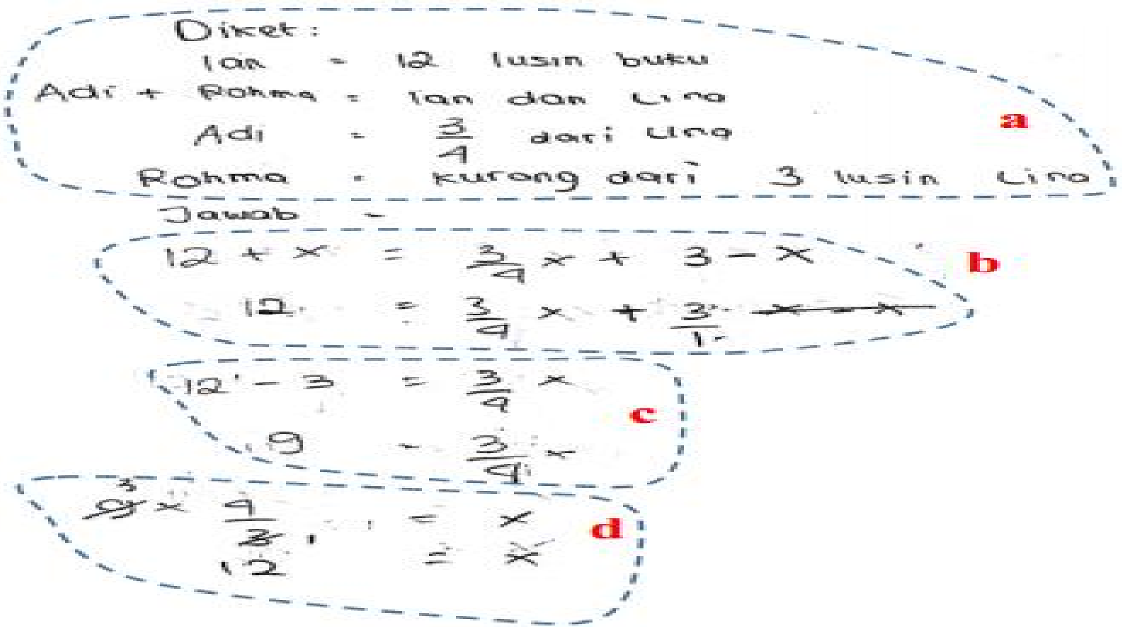

Figure 2. Student's results in understanding problems

The Figure 2-part a. is the student's first step in understanding the problems that exist in the narration problem. In this initial step, the student identified all the data facts in the problem, including the number of *Ian*'s books is 12 dozen, the number of Adi and Rohma's books is as many as *Ian* and Lina's books, the number of Adi's books is *¾* Lina's books, the number of Rohma's books is 3 dozen less than the number of Lina's books. At the stage of understanding the problem when determining the number of Rohma's books, the student cannot think reversibly in understanding the clause "the number of Rohma's books is 3 dozen less than the number of Lina's books", and this is basically equivalent to saying "the number of Rohma's books is the same as Lina's books less than 3 dozen". This shows the importance of students' reversible thinking in understanding problems. The importance of thinking reversibly in understanding mathematical problems to help students in solving the problem to the next stage is when students develop plans (Figure 2-part b) and problem-solving (Figure 2-part c. and Figure 2-part d). The inability of students to think reversibly when understanding problems will have an impact on misinterpretation, such as Figure 2-part b. which should be $x-3$. In addition, the figure *2b* shows that students are wrong in performing the principle of negation with *-x* - *x*, and the cancellation *-x* should be done with *+x* (Baroody et al., 2009; Maf'ulah et al., 2017; Ramful & Olive, 2008). Baroody et al (2009) show that the negation of the addition or subtraction operation is 0. The results of this preliminary study indicate that it is important to do further studies on how students' reversible thinking skills dealing with fraction problems presented in the narration questions. This article was aimed to (1) determine the aspect of student' reversible thinking ability dealing with fraction problems in the form of narration questions, and (2) determine the stages of students' reversible thinking in solving fraction problems.

METHODOLOGY

This study employs descriptive qualitative research. Qualitative research is a method for exploring and understanding social issues faced by a number of individuals or groups of people (Creswell, 2014). Descriptive research is a study that aims to describe existing phenomena, both natural and artificial phenomena (Sukmadinata, 2016). This study specifically describes and presents the aspects and stages of students' reversible thinking in solving the problems of fractions.

The subjects of this study were two students from the seventh grade of SMP N 3 Malang, Indonesia. The subjects were chosen purposively based on some criteria, i.e. (1) junior high school students aged between 13 to 15 years or students who are in the stage of abstract thinking, (2) the students have taken fraction material, (3) the students can communicate their thoughts both in speaking and writing, (4) the students are able to solve fraction problems.

The main data in this study were in the form of written answers and oral answers obtained from the test instruments and interviews. The fractional problems given to the students were about the comparison of the number of books owned by four students, in which the number of books owned by 1 out of 3 students has been determined; the other two students have the different number of books. Meanwhile, the question in this problem is how many books the fourth student has (see Figure 1). The data gathered from interviews were used data validation. In this study, the data validation used triangulation methods (Creswell, 2015), by re-checking the written answers, think aloud, and the results of interviews.

This research begins with conducting an initial study related to students' ability to communicate their thoughts both written and spoken. This initial study was conducted by giving a test to 87 students. The results of this test were discussed with the subject matter teacher to choose two students who are able to communicate their thoughts in writing and speaking and have the ability to solve fraction problems. There were 55 out of 87 students who were able to solve fraction problems; there were two students who could communicate their thoughts very well in writing and speaking. Furthermore, the two students were asked to explain the results of their work in writing at different times. The data yielded form students' work in writing and think aloud were described, and the data from interviews were used to validate the data that had been previously obtained, and then the analysis was carried out.

The data, in this study, were collected through written tests, observations, and interviews. This written test was used to determine the aspects and stages of students' reversible thinking in solving fraction problems. The movements and facial expressions of students are observed through the process of direct observation using video recording. Interviews were conducted to explore and confirm unclear information during the data collection. The data yielded from the written test, think aloud, and interviews were then analyzed to determine and describe the aspects and the stages of students' reversible thinking in solving fraction problems.

The data of this study were analyzed based on the steps of data analysis according to (Creswell, 2014) as follows.

1. Processing and preparing data to be analyzed, i.e., data from written test results, observations, and interview data.
2. Reading the entire data, i.e., building a general idea of the information obtained and reflecting on its meaning as a whole.
3. Analyzing in more detail by coding the data.
4. Applying the coding process to describe the types and themes to be analyzed.
5. Interpreting data, i.e., the researcher confirms whether the results obtained in the study are in line with or contradicted to previous information.

To verify the data in this study, a data triangulation technique (Creswell, 2014; Creswell, 2012) was used. Triangulation techniques are used to compare data obtained from the results of direct observation, interviews, and the results of students' written work in solving fraction problems.

RESULT AND DISCUSSION

First subject

The student's first step in solving a fraction problem was begun with reading the questions twice. Then, the student identified some facts in the data, such as the number of Ian's books, Adi's, and Rohma's. The identification results were, then, represented symbolically as shown in Figure 3 part 1.

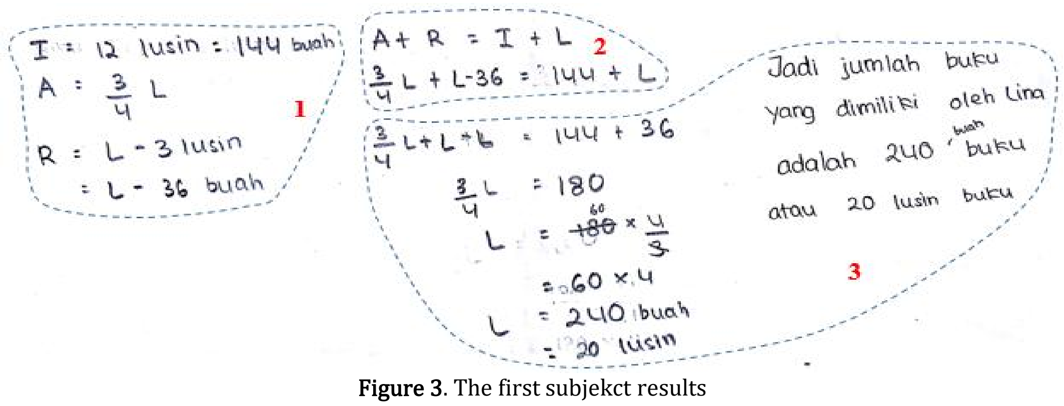

Figure 3. The first subjekct results
1. Symbolic representation by the first subject.
2. Symbolic representation by the first subject.
3. Symbolic representation by the first subject

The results of symbolic representation in identifying the data facts in the questions in Figure 3-part 1 show that the student converted the units of books from a dozen units to single-unit. The results of the conversion show that the student made a problem-solving plan. The results of the conversion were, then, substituted into the equations that the student succeeded in identifying, the number of Adi and Rohma's books equal to the number of Ian and Lina's books.

The equation that was successfully formed by the student in Figure 3 part 2 shows that the problem-solving action can already be done. Student's actions in solving problems were begun by grouping the L variable to one side of the equation and grouping the numbers of another side of the equation. The grouping of L variables shows that the student made a physical cancellation on one side of the equation so that it changes one side of the other equation. This change in equation shows that the equation remains in balance and is equivalent to the previous equation. At this stage, the student performed calculations so that the final result was obtained that the L value is 240 books or 20 dozen books. To find out more about how students think in solving fraction problems, researchers explored the data through interviews. The interview excerpts are presented as follows.

Q: *What are your steps in solving this problem?*

S1: *First, it is known that Ian has 12 dozen books or equals 144 pieces, the number of Adi and Rohma's books is the same as Ian's books and is added by the number of Lina's books, the number of Adi's books are ¾ from Lina's books, and Rohma's books are 3 dozen less than Lina's books.* (Students read the questions repeatedly).

Q: *What is the next step?*

S1: *I wrote each number of books* (the student wrote the results of the identification of the facts found in the *problem*). *First, Ian, Ian has 12 dozen, then Adi has ¾ of Lina's books or was written with ¾ L. Rohma has 3 dozen less than Lina's books* (the student confused and imagined the sentence)*, it means that the number of Lina's books was reduced by 3 dozen. Haaa ... yes ... 3 dozen or 36 pieces* (the student doubted the statement).

Q: *Next?*

S1: *Next, I wrote this equation* (the student pointed to the equation L + 1 = A + R)*. Adi ¾ L plus Rohma L - 36 = Ian 144 plus L as Lina. Next, the numbers are entered here and L is adjusted here* (the student grouped the numbers on one side and L as the variables grouped to another side).

The results of the interview show that there are several aspects of students' reversible thinking in solving fraction problems, as follows.

1. Converting a dozen units of books to single-unit of the book which indicates reciprocity.
2. Verbally revealing the form of denial in which it is verbally negation.
3. Negating one-sided variables or numbers so that it affects another side of the equation. This step occurs in addition (subtraction) and multiplication (division) operations. This action indicates reciprocity caused by negation.

The first subject had already shown reciprocity when understanding problems. Reciprocity occurred when the student understood that 12 dozen books are the same as 144 books, and vice versa, 144 books are equal to 12 dozen books. This is in line with Ramful (2014) who stated that reciprocity can be identified through logical equation, if A = B, then B = A. Reciprocity also occurred when the student understood the phrase *'....3 dozen less than the number of Lina's books'* is equivalent to *'the number of Lina's books minus three dozen books'*, and this is due to the verbal negation in the clause *'the number of Rohma's books is 3 dozen less than the number of Lina's books'*. When understanding the clause *'the number of Rohma's books is 3 dozen less than the number of Lina's books'*, the student experienced perplexity and tried to overcome it by reading the questions repeatedly. In this process, the student tried to reverse the statement *'3 dozen less than the number of Lina's books'* to *'the number of Lina's book minus three dozen'.* This is identified as negation (Nordmeyer & Frank, 2014). Reciprocity caused by negation also occurred when the student carried out problem-solving actions, in which grouping L variable into one side of the equation and grouping numbers into another side of the equation. This grouping activity is termed by the student with *'moving segments'* in an equation. This moving segment is based on the existence of negation or negation with $+L - L = 0$ (Baroody et al., 2009).

'*Moving segments'* is a term which is commonly used by Indonesian students. This *'moving segments'* is based on inverse operation, as in this reasoning $+L$ (on the left side) becomes $-L$ (on the right side). An inversion is an example form that produces a concept that remains the same in some algebraic structures (Wasserman, 2016), i.e., if $+L$ is added to $-L$, it is equal 0, and this is the result of negation 0 (Baroody et al., 2009). Maf'ulah et al. (2017) also show that the displacement of $+8$ (on the right side) to -8 (on the left side) of an equation indicates that it is a reversible aspect of thinking in the form of negation. The result of this negation will cause the condition to return to equal as the initial equation, and equality is the principle of reciprocity (Saparwadi et al., 2019; Martin et al., 2016).

In general, the stages of the student's, the first subject, reversible thinking in solving fraction problems based on the aspects of reversible thinking can be seen in Figure 4 below

Second Subject

The problem-solving stage of the second subject is almost similar to the first subject, which was begun by identifying all the facts of the data in the problem. The next step, the student represented the data that has been identified symbolically. While writing symbolic representation, the student prepared a completion plan that was marked by the process of converting a dozen units into single-unit of the book. The unit conversion results in the number of books represented symbolically which can be seen in Figure 5 part 1 below.

The arrangement of the problem-solving plan in Figure 5 part 1 shows that the student took action to solve the problem. The complete problem-solving activities by the second subject can be seen in Figure 5 part 2. The problem-solving activities by the second subject are almost similar to the first subject. At this stage, the student, first, simplified the equation through the calculation process. Next, the student grouped L variables into one side of the equation and grouped the numbers on another side of the equation. The grouping process was, then, continued with the calculation process until the L value of 240 books was found, and with L refers to Lina. To show check the validity of the results obtained, the student conducted a reflection by substituting the L value obtained into the initial equation. The results of the student's reflection can be seen in Figure 5 part 3.

To find out the stages of students' reversible thinking in solving problems, an in-depth interview was conducted to explore the aspects that students use in thinking. The followings are the interview excerpts.

Q: Would you please to share what are your steps in solving this problem?

S2: First, Ian has 12 dozen books. Then it is written 12 dozen = 144. Next, there is a question of the number of Adi and Rohma's books which is equal to the number of Ian and Lina's books. It means A + R = I + L. Here, it is stated that for example, the number of Adi's books is ¾ of the number of Lina's. Meanwhile, the number of Rohma's books is 3 dozen less than Lina's. It means A = ¾ L, R = L - 3 dozen = L - 36. I wrote it down like this to make it easier.

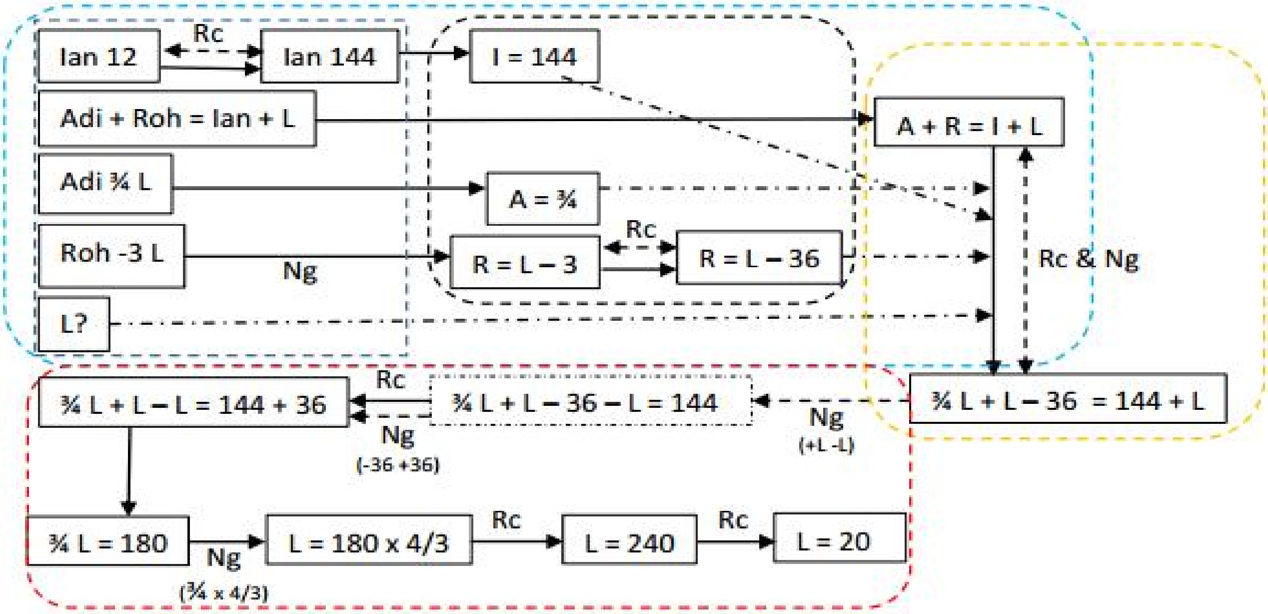

Figure 4. The stages of the first subject's reversible thinking

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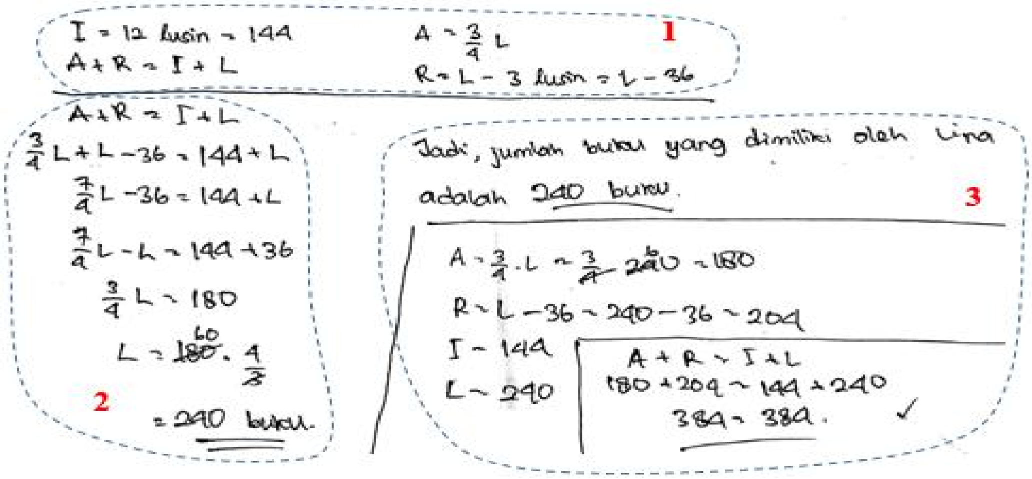

Figure 5. The second subjekct results

1. The activity of understanding the problem by the second subject.
2. The activity of problem-solving by the second subject.
3. The results checking activity by the second subject.

Q: What did you do then?

S2: Here is the equation: A + R = I + L. A = ¾ L and R = L - 36, and ¾ L + L - 36 = I + L or 144 + L. ¾ L + L = 7/4 L, so that it becomes 7/4 L - 36 = 144 + L. This L (while pointing to the left side*) is moved, so that it becomes 7/4 L - L = 144 + 36. Then it becomes ¾ L = 180, then L = 180 multiplied by 4/3 equals 240 books.*

Q: How could you think that the results you get are correct?

S2: Here is the proof (pointing the operation of A + R = I + L)*. A = ¾ L = ¾ x 240 the result is 180, while R = L - 36 is 240 - 36 and the result is added to 180* (pointing to the result of A = ¾ L)*. Next, I + L = 144 + 240. If all symbols have been found their values, then the operation is done. If the results of the left and right sides are correct, it means that the results are correct.*

The results of this interview show that there are several aspects of students' reversible thinking in solving fraction problems as follows.

1. Converting a dozen units of books to single-unit of the book which indicates reciprocity.
2. Verbally revealing the form of denial in which it is verbally negation.
3. Negating one-sided variables or numbers so that they affect another side of the equation. This step occurs in addition (subtraction) and multiplication (division)

operations. This action indicates reciprocity caused by negation.

4. Reviewing the results obtained by returning them to the initial equation through the process of substituting the results found. This is a process of anticipation and review. This activity occurs because of the verbal negation of students.
5. The results of this review cause reciprocity.

The stages of the student's, second subject, reversible thinking in solving problems were also begun with reciprocity, which is to understand 12 dozen = 144 units of books and equivalent to 144 units of books = 12 dozen books which are the number of Ian's books. Reciprocity is also seen when the student understood *L - 3 = L - 36* or *L - 36* units of book = L - 3 dozen books, and this is equivalent to 3 dozen less than L, where L is the number of Lina's books. When the student reversing his thought about *'....3 dozen less than the number of Lina's books'* into the number of Lina's books reduced by 3 dozen (*L - 3*), according to (Nordmeyer & Frank, 2014), this is a negation identified from verbal denial (Hanfstingl et al., 2019; Lourenço, 2019). Negation and reciprocity also occurred simultaneously when the student changed *¾ L + L - 36 = 144 + L* to *¾ L + L - L = 144 + 36*. The negation result in this process is 0 (Baroody et al., 2009) for addition (reduction) operations. While the student's understanding of *¾ L + L - L = ¾ L* is one of the principles of the inverse (Norton & Wilkins, 2012; Baroody et al., 2009; Robinson & LeFevre, 2012).

Negation and reciprocity continuously occur simultaneously as long as students complete equality in formal matters (Inhelder & Piaget, 1958). Basically, Piaget's reversible thinking identification in students can also be known from their arguments about the problem they face. When students understand the problem based on the argument that the results of solving the problem remain the same despite being redone (Hackenberg, 2010; Ramful, 2014, 2015; Simon et al., 2016), this shows the existence of identity (Baroody et al., 2009) from students' reversible thinking. But in this case, students do not show the argument, and students prefer to show the results, so it needs to be proven again.

The aspects that underlie the students to re-prove the results that have been obtained to the initial equation are based on the existence of uncertainty of the results obtained, and Nordmeyer & Frank (2014) call it inferential negation or the negation of belief. This shows that negation is one of the steps students take to overcome the perplexity that occurs in them. According to Goodwin et al. (2009), re-proves or verification process whether or not the result obtained is the solution to the initial equation is called 'back substitution' phase. However, the substitution phase is not defined in detail, so Sangwin & Jones (2017) use the term 'reversible processes' which are based on the idea that mathematics involves a lot of symbolic reversibility, such as formal mathematics. This shows that negation as an aspect of reversible thinking has an important role in solving problems, especially in mathematical problems.

In general, the stages of the student's, the second subject, reversible thinking in solving fraction problems based on the aspects of reversible thinking can be seen in Figure 6 below.

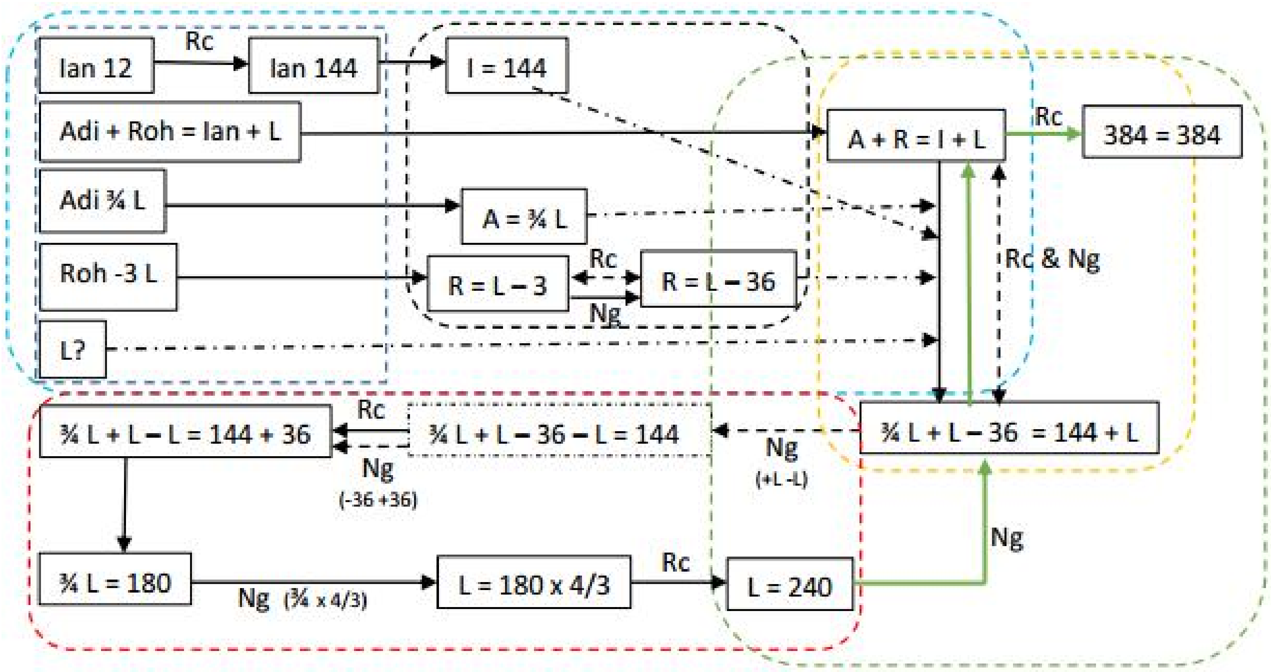

Figure 6. The stages of the second subject's reversible thinking

Table 1. The description of the code of reversible thinking process of the subject

Term	Code
The number of Ian's books is 12 dozen	Ian 12
The number of Ian's books is 144 books	Ian 144
The number of Adi's book is ¾ of Lina's book	Adi ¾ L
The number of Rohma's book 3 dozen less than Lina's book	Roh -3 L
How many books does Lina have?	L?
Adi	A
Rohma	R
Ian	I
Lina	L
Negation	Ng
Reciprocity	Rc
Identifying the facts of data and problem	
The representation of the problem	
Understanding the problem	
Making a solving action plan	
Doing problem-solving	
re-checking	

In general, the difference between the findings of the first subject and the second subject lies in the verification process or re-checking. The second subject has an argument that the results obtained need to be verified or re-checked to the initial conditions and this is the principle of negation of thinking reversibly (Saparwadi et al., 2019). Meanwhile, the first subject did not show this and only stated that the results had been found. The findings of this study are quite different from the findings of Maf'ulah et al. (2017) who concluded that the aspect of students' reversible thinking in solving problems was negation and reciprocity based on students' mathematical operations. Maf'ulah et al. (2017) do not identify that there is a verbal negation, the negation of beliefs and negation can lead to reciprocity. In addition, Tunç-Pekkan (2015) also does not identify any aspect of reversible thinking in detail, such as negation, reciprocity, and inverse.

For this reason, several findings from this study dealing with the aspects that cause students to think reversibly are (1) unit conversion which results in the aspect of reciprocity; (2) Negation occurs, one of them because of the inverse effect; (3) Negation will cause re-verification or re-checking results to the initial condition (Saparwadi et al., 2019); (4) Negation will cause reciprocity, especially in algebraic operations; (5) Perplexity will cause negation, and (6) Reciprocity will overcome perplexity. In addition, in this study, it was found the different stages in reversible thinking from both subjects. The stages of students' reversible thinking in solving problems begins with understanding the problem, arranging the plan of action, implementing the plan, and checking the results obtained. The difference stage of the two subjects lies on the stage of re-checking the results obtained. The first subject checked the problem by merely believing that the resulted answer is correct and does not re-check to the initial physical condition, so the negation of reciprocity cannot be identified at this stage. It is different from the second subject who rechecked the result with action so that negation and reciprocity can be identified.

CONCLUSION

The aspects of students' reversible thinking, the subjects, are negation, inverse, and reciprocity. One of the negations is due to the inverse effect. Negation and reciprocity occur simultaneously in formal problems, and reciprocity occurs due to the negation. In addition, reciprocity occurs when students convert units. Meanwhile, negation occurs through the process of moving a variable part (number) that produces an element of identity. Based on the stages of reversible thinking in the two subjects, negation is very important to overcome perplexity in solving problems, and reversible thinking can occur in students who are solving problems, especially in the field of mathematics.

ACKNOWLEDGMENTS

This work was supported by Universitas Negeri Malang dan LPDP (Endowment Fund for Education) and The Ministry of Research, Technology, and Higher Education of Indoesia.

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